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Constant bed-length folding: three-dimensional geometrical implications

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Abstract—The folding of a sheet without stretching of lines within it is referred to, in the language of differential geometry, as an isometric bending. As can be readily verified by flexing sheets of paper, the no-stretch condition imposes important restrictions on the curvature changes which points on the sheet can undergo during folding. These constraints are embodied in Gauss's Theorema Egregium which states that "the total curvature (equal to the product of the two principal curvatures) at any point remains invariant under isometric bendings". It follows from this theorem that there is a limited range of folded geometries that an initially planar non-stretching sheet can adopt. These developable surfaces, which include cylindrical and conical folds, have the property that points of equal dip and strike of the surface are arranged in straight lines.

This property allows a simple check to be made of the validity of the constant bed-length assumption in the case of natural fold structures. For any fold represented by structure contours, points of equal strike on the structure are linked by isotrend lines which will be straight if the structure is developable. Curved isotrend lines indicate that the structure has a geometry incompatible with the constant bed-length model.

The patterns of isotrend lines constructed for a fold help to indicate parts of the structure where layer stretching or faulting is likely. On the other hand, patterns consisting of straight isotrend lines can be used for the prediction of the structure in adjacent areas. Isotrend analyses to date suggest that bed-length balancing of cross-sections is generally an invalid procedure.

INTRODUCTION

DURING the analysis of a wide variety of tectonic structures (thrusts, normal faults, folds) the assumption is frequently made that the length of deformed beds, measured in cross-section, equals the length of the beds in the pre-deformation state. Although the main justification for making this assumption is usually one of analytical convenience, the property of constancy of bed-length is implied in the flexural folding and neutral surface folding models. In natural folds, the condition of constant bed-length will be approached in the case of parallel (class 1B) folds produced in strongly anisotropic layered sequences or in thin beds with high relative competence.

The application of this assumption to the situation involving several beds in a cross-section through a structure imposes severe limitations on the possible two-dimensional geometry of the structure. This is the basis of the well-known technique of cross-section balancing.

This paper examines the three-dimensional implications of the constant bed-length folding model to discover the range of possible fold forms which can result from this mode of deformation. Once these feasible fold geometries are defined from a theoretical standpoint, attempts are made to devise forms of analysis which allow these special fold geometries to be recognized amongst natural structures. This, in turn, allows conclusions to be drawn regarding the presence or absence of bed stretching associated with particular natural fold structures.

ISOMETRIC DEFORMATION

The deformation involved when a sheet is bent without the stretching of lines contained within its surface is referred to, in the language of differential geometry, as *isometric bending*. Such deformation is characterized also by the conservation of the size of angles between pairs of lines and by strain ellipses in the plane of the folded surface which are circles of unit radius (Fig. 1).

A remarkable property of isometric bending has to do with the way curvature at points on a surface is affected by the folding. Although the curvature of lines will in general become changed, Gauss's Theorema Egregium

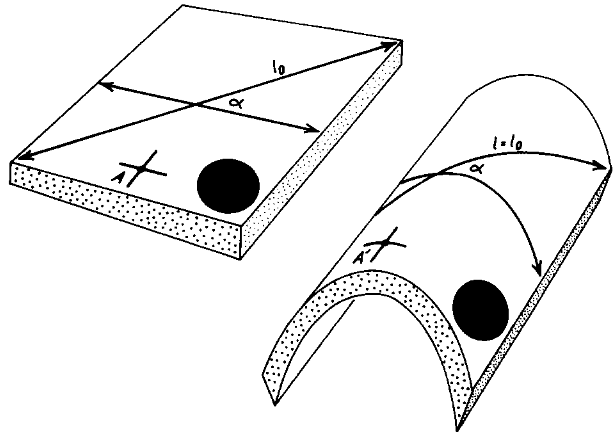

Fig. 1. Isometric bending. The folded surface is one of zero sectional strain. The total or Gaussian curvature at points (e.g. A and A') does not change.

states that the "total curvature K (the product of the two principal curvatures) is invariant under isometric deformation" (see Hilbert & Cohn-Vossen 1952, Suppe 1985).

Gauss's theorem has important implications for the geometry of folded surfaces that can possibly develop from the isometric bending of initially planar surfaces. Since a flat sheet has zero total curvature at all of its points, so too must the surface which develops by folding. For example in Fig. 1 the point A has zero total curvature ($K = 0$) before and after folding. This means that domes and basins are excluded as possiblities since they are made up of points with positive K. Similarly saddles and shoe horns, which possess negative K, are fold shapes that are incompatible with the isometric deformation of a plane.

The fold shapes that can be formed from isometric bending of a flat sheet consist solely of points of zero total curvature and belong to a class of surfaces called *developable surfaces* (Hilbert & Cohn-Vossen 1952).

DEVELOPABLE FOLDS

As can be readily demonstrated by flexing a sheet of paper, a limited range of fold forms is capable of being produced by isometric bending. All developable surfaces are ruled surfaces which means that straight lines (generators) can be drawn on them and have the additional property that adjacent generators, if extended far enough, will intersect one another (Hilbert & Cohn-Vossen 1952). In the general case of a developable surface, successive points of intersection of generators define a curve in space (Fig. 2a). The curve can consist of a single point (Fig. 2b) as in the case of conically-folded surfaces. If this point is situated at infinite distance the surface becomes cylindrical.

From a geological point of view this curve made up of the intersection points of generators is important because it marks the maximum extent of the isometrically folded sheet. In other words, the isometric folding model is not capable of explaining the structure of an infinitely-extensive sheet; only of compartments within it bounded by some form of discontinuity.

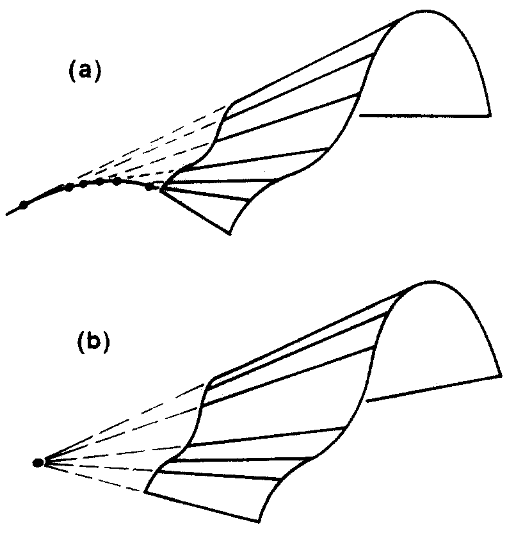

Fig. 2. Developable surfaces. (a) General case in which a curve is formed from the intersection of generators; (b) special case of a conical surface.

At all points lying along any generator line of a developable surface, unlike other classes of ruled surfaces, the orientation of the surface is constant. This characteristic proves to be of great value for the identification of isometric folds.

DISTINGUISHING DEVELOPABLE AND NON-DEVELOPABLE FOLDS

In the case of outcropping folded surfaces (Fig. 3), the absence or presence of double-curvature can be used for the recognition of developable and non-developable folds, respectively. The examples in Fig. 3(a) show folded surfaces upon which straight lines could be drawn. Their detailed geometry is close to cylindrical and the folds are therefore developable; that is, they have a geometry compatible with isometric folding. The developable geometry does not of course prove that the folds formed by that mechanism; only that their geometry is consistent with that mechanism. In contrast with these structures, it is clear that straight lines cannot be drawn on the folds exhibited in Fig. 3(b). We can conclude that the latter could therefore not have developed from isometric bending of planar layers.

Situations in which the complete folded surface is exposed are of course rare. More frequently the geometrical characteristics have to be deduced from collections of data consisting of the orientations of the surface measured at outcrops which are small in relation to the size of the fold structure.When such data (normals to bedding) are analysed stereographically, developable folds should produce linear arrays of poles whereas surfaces with double curvature will be expected to yield more disperse clouds (Hilbert & Cohn-Vossen 1952, Rech 1977).

However there are two potential difficulties with the practical application of the stereographic method. There is the problem of the natural statistical dispersion which will always be present in natural orientation data and which will make the distinction of the developable and non-developable forms less clear. The distinction may also be blurred because of the fact that any data set collected from a sizeable area is likely to originate not from a single surface but from several.

A more fruitful approach analyses the morphology of surfaces represented by structure contours. Firstly a general assessment of the curvatures present can be obtained by visual inspection of the structure contour map. The type of pattern exhibited by the contours is indicative of the total curvature (Fig. 4). Surfaces with positive curvature such as domes and basins display closed elliptical patterns (Fig. 4a). Folds consisting of points of negative total curvature (e.g. saddles and shoe horns) show hyperbolic patterns (Fig. 4b). On the other

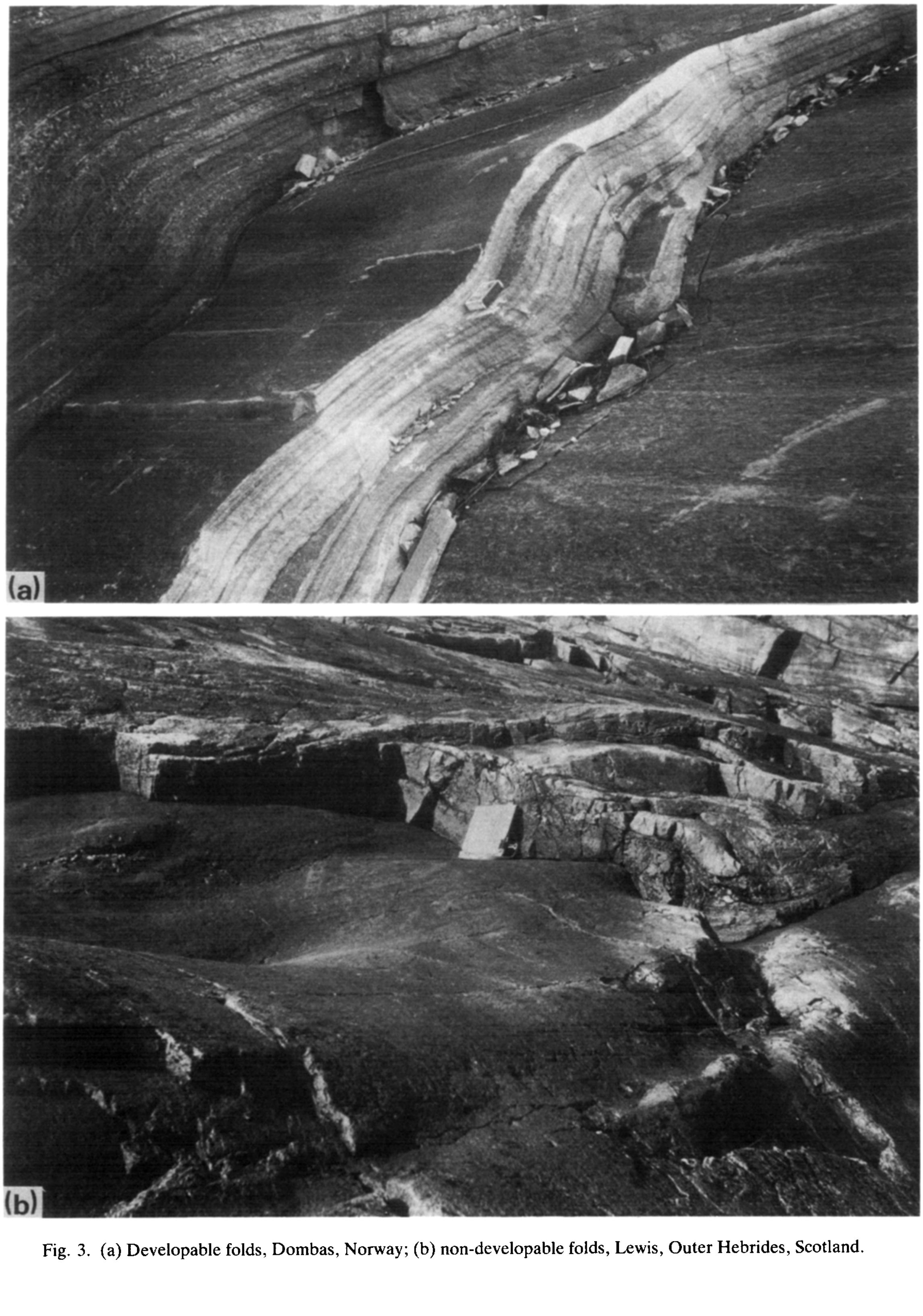

Fig. 3. (a) Developable folds, Dombas, Norway; (b) non-developable folds, Lewis, Outer Hebrides, Scotland.

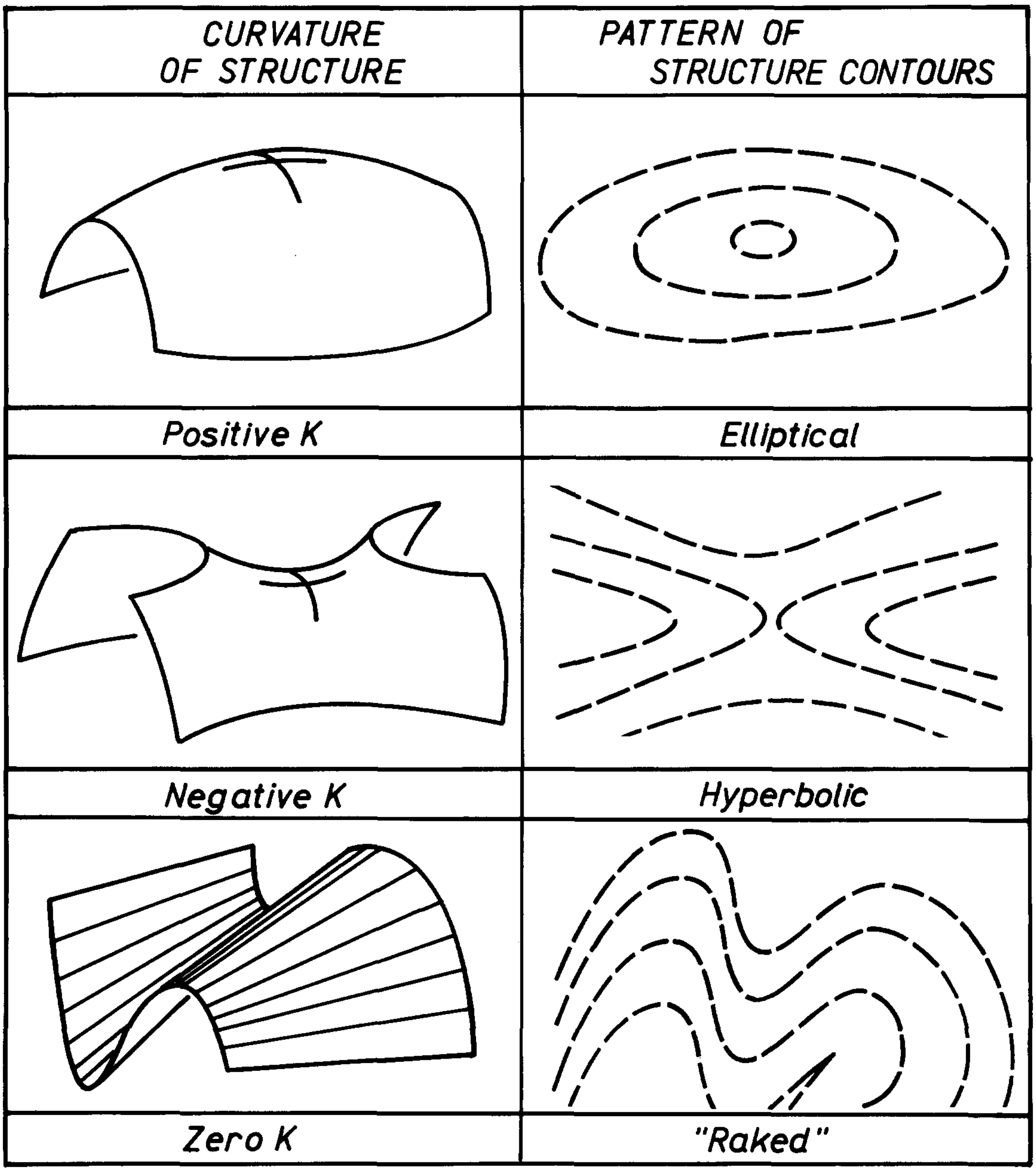

Fig. 4. The relationship between surface curvature and structure contour patterns.

hand, developable folds (surfaces with zero total curvature at every point) yield 'raked' patterns, i.e. the configuration of contours that could be swept out by the spikes of a rake or comb (Fig. 4c).

Some of the above features, such as the elliptical closure of contours, occur only at specific positions on the map; the exact position depending on the orientation of the surface relative to the horizontal. To analyse the curvatures at all points on a map involves drawing isotrend lines. The proposed method utilizes the property that developable surfaces are made up of straight lines or generators along which the strike and dip of the tangent planes to the surface are constant (Fig. 5).

On the map, points are sought on successive contour lines where the trends of the contours are equal (Fig. 6). These points are located using the method for constructing dip isogons on fold profiles (Ramsay 1967, p. 363, Ragan 1985, p. 207). These points are then joined to form lines of constant contour trend or isotrend lines.

Conclusions can be drawn regarding the morphology of the folded surface on the basis of the pattern shown by the isotrend lines on a map. If the surface developed by isometric bending, the isotrend lines will be straight lines. A further property of developable surfaces is that the spacing of the structure contours, measured along the isotrend lines, is constant. These two characteristics are not shared by non-developable surfaces.

For developable structures the isotrend line method actually determines the set of generators corresponding to the surface being analysed. In such cases, the generators, which plunge in the direction of the isotrend lines and plunge at an angle governed by the spacing of the structure contours, are the appropriate axes for the projection of dips, for example on to cross-sections. This

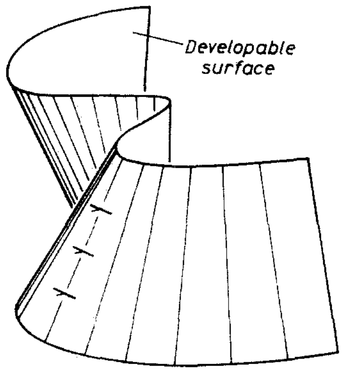

Fig. 5. Developable surface. At all points on a generator the surface has the same dip and strike.

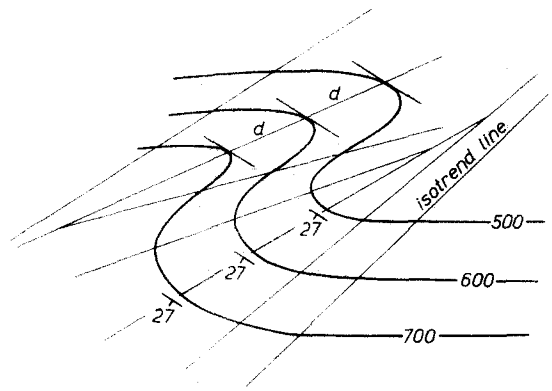

Fig. 6. Isotrend line pattern of a developable surface. Isotrend lines are straight and spacing, *d*, is constant.

method can be considered a generalization of the notion of 'down plunge' projection technique (see De Paor 1988 for a comprehensive bibliography).

Isometrically-folded layers are, with the exception of cylindrical surfaces, of limited areal extent. The analysis of the geometry by means of isotrend lines allows the boundaries of the sheet to be defined. The straight line isotrend patterns which are characteristic of constant bed-length folds usually exhibit convergence of adjacent isotrend lines. If projected far enough, these lines will meet in points which together define a line in space marking the edge of the isometrically-folded sheet (Figs. 2 and 6).

To the author's knowledge, this is the first time that isostrike lines have been used for the analysis of folds. Agterberg (1964) has analysed the structure of a folded region in N. Italy by means of constructed 'iso-lines of strike' but the latter have a different significance to the present isostrike lines as they refer to the mean local strike of several surfaces.

EXAMPLES OF ISOTREND ANALYSES

The structure contour map of the Candeias Oilfield, Brazil (Dobrin 1977), can be used as an example of this type of analysis. This map (Fig. 7) shows a number of features which are at odds with the constant bed-length folding model. In the first place, the existence of elliptical and hyperbolic patterns of structure contours indicates non-zero total curvature. In addition, the constructed isotrend lines possess a curved shape and furthermore the spacing of the isotrend lines along them is not uniform. It can be concluded that the development of the fold structures in this area involved a component of bed-stretching and/or contraction.

In the structure contour map (Fig. 8) of the Red Wash Field, Utah (Dobrin 1977), isotrend analysis of the NW part of the map reveals a geometry consistent with the constant bed-length folding mechanism, i.e. the isotrend lines are approximately straight and the spacing of the contours along them is fairly uniform. Cases such as this allow the possibility of extrapolating the geometry of the folds into adjacent regions. By extending the isotrend lines to the northwest (Fig. 8), the shape of the structure contours can be predicted.

In other regions of the map (Fig. 8), strongly curved isotrend lines indicate that locally the folding process must have involved a component of distortion in the bedding planes. Although the present technique does not permit quantitative estimation of the bed strain, these regions of bed stretching, contraction or area change could be favoured sites of strain-induced porosity or permeability changes (Gerla 1987). It is suggested therefore that this simple form of analysis could have practical benefits for exploration.

The points of convergence of neighbouring isotrend lines theoretically define one edge of the folded sheet. In practice these points suggest either the existence of faults or of an area where the conditions of the model are being violated, e.g. where bed stretching has taken place. An example of this is the map in Fig. 9 of the

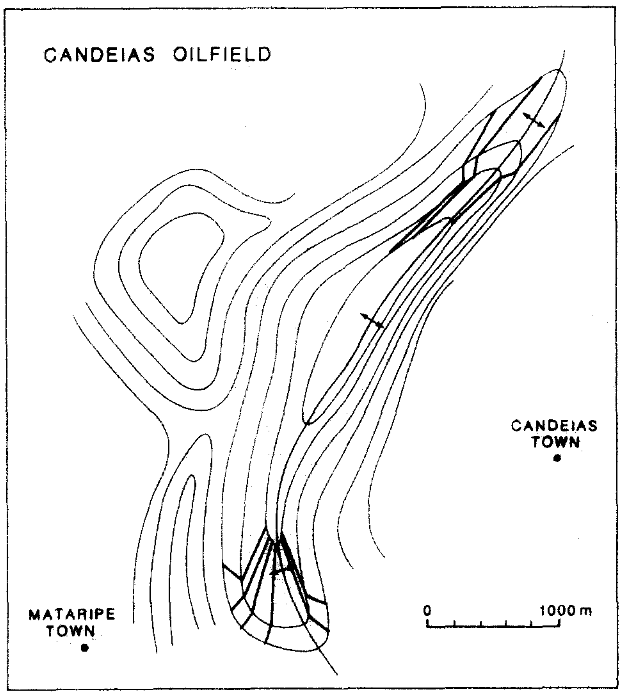

Fig. 7. Seismic structure map for the Candeias field in Brazil (after Dobrin 1977). Isotrend lines are shown as thick lines.

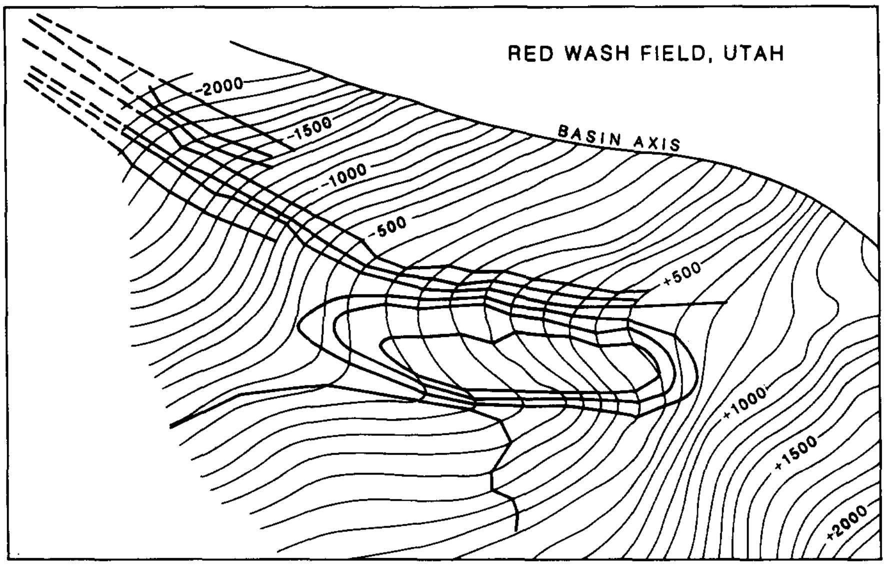

Fig. 8. Isotrend analysis of the Red Wash Field, Utah. Isotrend lines; thick lines.

Quitman Field, Wood County, Texas (De Sitter 1958), where the isotrend line patterns seem to converge towards the sites of mapped faults. This suggests that isotrend line patterns have a potential for the prediction of fault structures.

RESTORING DEVELOPABLE STRUCTURES

Compared to other models of folding, isometric folds are particularly amenable to restoration. During the process of isometric folding the attitudes of the limbs and of lines within the folded surface change solely as a result of a simple rotation about the local generator on the surface. As no strain exists in the surface being folded, restoration involves simply the 'unrolling' of the surface back to a plane. Local rotations of one part of the sheet about the local generator will permit the relative orientations of the two adjacent parts of the sheet to be determined.

However there are two practical difficulties that need to be overcome before absolute orientations of lines within the sheet can be determined. Firstly, the restoration by rotation has to be carried out for the whole sheet since the change of orientation of part of the sheet depends on the integrated rotations involved in other points in the sheet. Secondly, as can be visualized in the example of a wrinkled table-cloth containing a single crumple with the shape of a half cone, the unrolled configuration depends on the choice of the starting-point for the unrolling. In other words, the restoration requires the recognition of a part of the sheet which is still in its original orientation.

Two-dimensional restoration of the structures displayed in cross-sections involving constant bed-length folds, and subsequent estimates of shortening, are made invalid by the fact that straight lines generally fold to give non-plane curves. For developable folds which are cylindrical this has been discussed in detail by McCoss (1988). Conversely, curves representing a planar section

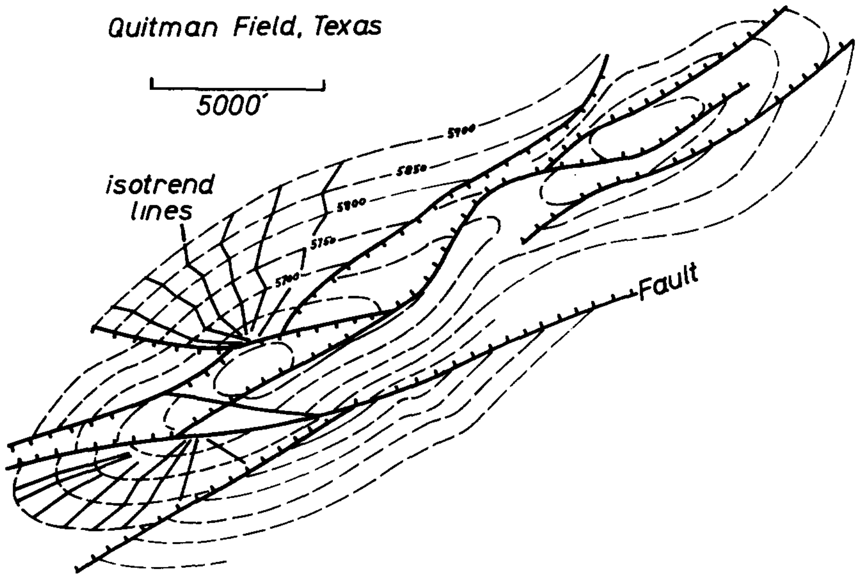

Fig. 9. Isotrend analysis of the Quitman Field, Texas (De Sitter 1958). Isotrend lines converge towards mapped faults.

through a developable surface generally will correspond to curved lines in the pre-folding state.

USING THE MODEL AS A BASIS OF CONTOURING

So far the method of isotrend analysis has been used to determine the geometry of folded surfaces and to draw inferences regarding the amount of bed stretching from structure contour maps. The analysis assumes that a reliable structure contour map is available at the outset. However it is appreciated that a reliable structure map is a goal, not a starting point for most structural analyses. It could be suggested that if the isotrend lines on a given map are not straight then perhaps it is the structure contour map that needs amending!

Although the practicalities have not yet been fully worked out, the model of isometric folding could be used as a basis of a method of contouring data consisting of spot elevations of a folded surface. On the assumption of this isometric deformation, the contours would have to meet, as closely as possible, the criterion of constant spacing along straight isotrend lines.

Although isometric folding is seen as an ideal, to which actual folds may approach to greater or lesser degrees, such a contouring procedure is considered preferable to others based on arbitrary mathematical surface-fitting routines.

CONCLUSIONS AND DISCUSSION

Isotrend line analysis is capable of detecting fold surface morphologies which require bed stretching for their development. This simple technique can be used to highlight areas of probable bed-plane strain or faulting.

Caution is advocated when drawing conclusions from isotrend line patterns. Although isometric deformation implies developable fold geometries, the converse is not necessarily true. Cylindrical folds are examples of developable surfaces that could form from a variety of mechanisms including those of a non-isometric nature. Also a homogeneous flattening strain imposed on any developable surface will modify its shape but not its developable character. Although only a few structure maps have so far been subjected to the analysis described above, it is already apparent that bed-strain is normally involved in folding. In this respect it is interesting to note that hydrocarbon entrapping fold structures with 'closure' cannot form by constant bed-length folding. In a strict sense, line-length balancing of cross-sections is a theoretically unsound technique even in cases where the folds involved are isometric.

Although the isometric folding model is an idealized one, its predictions may be of relevance, at least in a qualitative way, to sequences in which a parallel folding style is developed. The model requires a careful consideration of curvature which has important implications for the problems of refolded folds (Stauffer 1988, Lisle *et al.* 1990), salt domes, subducting plates (Bayly 1982) and the nature of the terminations of folds (Wilson 1967, Webb & Lawrence 1986).

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